PMM U.S.S.R., Vol.50, No.6, pp.789-799, 1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

THE CONTACT PROBLEM OF ELASTICITY THEORY FOR BODIES WITH CRACKS*

I.I. KUDISH

The plane contact problem of a stamp impressed into an elastic half-plane containing arbitrarily arranged rectilinear subsurface cracks is formulated and investigated by asymptotic methods. Partial or total overlapping of the crack edges is assumed. The problem reduces to a system of linear singular integrodifferential equations with side conditions in the form of equalities and inequalities. An analytic solution of the problem is obtained in the form of asymptotic power series /1/ in the relative dimension of the greatest crack. Dependences of the first terms of the asymptotic expansions of the desired functions on the mutual location of the cracks and the contact domains, the pressure and friction stress distributions, and the crack size and orientation are determined. Numerical results are presented.

Analysis of the influence of the stress-free boundary of the half-plane on the state of stress and strain of the elastic material near the cracks is presented in /2, 3/. The problem of a crack in an elastic plane whose edges overlap partially is also examined in /3/ by numerical methods.

1. Formulation of the problem. We consider a plane problem on the frictionless interaction of a stamp with the base $z = f(x)$ and a crack-weakened elastic half-plane (the problem taking friction into account is investigated in Sect.3). It is assumed that there are N rectilinear subsurface cracks in the half-plane, on whose edges there is no friction.

In dimensionless variables

$$\{x', \tau', a, c, x_k^{\circ\prime}, y_k^{\circ\prime}\} = \{x, \tau, x_i, x_e, x_k^\circ, y_k^\circ\}/b_0, \ \{q', p_k'\} = \{q, p_k\}/q_0, \ \{x_k', t'\} = \{x_k, t\}/l_k, \ \{v_k', u_k'\} = \{v_k, u_k\}/v_k^\circ \tag{1.1}$$

$$f'(x') = \frac{\pi E'}{2P} f(x), \quad \delta^{\circ\prime} = \frac{\pi E'}{2P}\delta^\circ - \ln\frac{1}{b_0}; \quad v_k^\circ = \frac{4q_0 l_k}{E'}, \quad \delta_k = \frac{l_k}{b_0}$$

the problem reduces to a system of equations with additional conditions in the form the equalities and inequalities /4/ (the primes are omitted)

$$f(x) + \frac{2}{\pi}\int_a^c q(t)\ln\frac{1}{|x-t|}\,dt - \frac{2}{\pi}\sum_{k=1}^N \delta_k \int_{-1}^1 \{v_k'(t) W_k^r(t,x) - u_k'(t) W_k^i(t,x)\}\,dt = \delta^\circ, \quad \int_a^c q(t)\,dt = \frac{\pi}{2} \tag{1.2}$$

$$W_k(t,x) = ie^{-i\alpha_k}\frac{\bar{T}_k - T_k}{\bar{T}_k - x}, \quad W_k^r = \operatorname{Re} W_k, \quad W_k^i = \operatorname{Im} W_k \tag{1.3}$$

$$\int_{-1}^1 \frac{v_n'(t)\,dt}{t-x_n} + \sum_{k=1}^N \delta_k \int_{-1}^1 \{v_k'(t) U_{nk}^r(t,x_n) - u_k'(t) V_{nk}^r(t,x_n)\}\,dt = \pi p_n(x_n) - \int_a^c q(\tau) D_n^r(\tau, x_n)\,d\tau \tag{1.4}$$

$$\int_{-1}^1 \frac{u_n'(t)\,dt}{t-x_n} + \sum_{k=1}^N \delta_k \int_{-1}^1 \{-u_k'(t) V_{nk}^i(t,x_n) + v_k'(t) U_{nk}^i(t,x_n)\}\,dt = -\int_a^c q(\tau) D_n^i(\tau, x_n)\,d\tau$$

*Prikl.Matem.Mekhan., 50,6,1020-1033,1986

$$U_{nk}=\overline{R_{nk}+S_{nk}},\quad V_{nk}=\overline{i(R_{nk}-S_{nk})} \tag{1.5}$$

$$U^r_{nk}=\operatorname{Re}U_{nk},\quad U^i_{nk}=\operatorname{Im}U_{nk},\quad V^r_{nk}=\operatorname{Re}V_{nk},\quad V^i_{nk}=\operatorname{Im}V_{nk}$$

$$D_n{}^r=\operatorname{Re}\overline{D}_n,\quad D_n{}^i=\operatorname{Im}\overline{D}_n$$

$$R_{nk}(t,x_n)=(1-\delta_{nk})K_{nk}(t,x_n)+\frac{e^{i\alpha_k}}{2}\left\{\frac{1}{X_n-\overline{T}_k}+\frac{e^{-2i\alpha_n}}{\overline{X}_n-T_k}+\right.$$
$$\left.(\overline{T}_k-T_k)\left[\frac{1+e^{-2i\alpha_n}}{(\overline{X}_n-T_k)^2}-\frac{2e^{-2i\alpha_n}(X_n-T_k)}{(\overline{X}_n-T_k)^3}\right]\right\}$$

$$S_{nk}(t,x_n)=(1-\delta_{nk})L_{nk}(t,x_n)+\frac{e^{-i\alpha_k}}{2}\left[\frac{T_k-\overline{T}_k}{(\overline{X}_n-\overline{T}_k)^2}+\right.$$
$$\left.\frac{2}{\overline{X}_n-T_k}-e^{-2i\alpha_n}\frac{X_n-T_k}{(\overline{X}_n-T_k)^2}\right]$$

$$K_{nk}(t,x_n)=\frac{e^{i\alpha_k}}{2}\left(\frac{1}{T_k-X_n}+\frac{e^{-2i\alpha_n}}{\overline{T}_k-\overline{X}_n}\right)$$

$$L_{nk}(t,x_n)=\frac{e^{-i\alpha_k}}{2}\left[\frac{1}{\overline{T}_k-\overline{X}_n}-\frac{T_k-X_n}{(\overline{T}_k-\overline{X}_n)^2}e^{-2i\alpha_n}\right]$$

$$D_n(\tau,x_n)=\frac{i}{2}\left[-\frac{1}{\tau-X_n}+\frac{1}{\tau-\overline{X}_n}-\frac{e^{-2i\alpha_n}(\overline{X}_n-X_n)}{(\tau-\overline{X}_n)^2}\right]$$

$$X_n=\delta_n x_n e^{i\alpha_n}+z_n^\circ,\quad T_k=\delta_k t e^{i\alpha_k}+z_k^\circ,\quad z_k^\circ=x_k^\circ+iy_k^\circ \tag{1.6}$$

$$p_n(x_n)=0,\quad v_n(x_n)>0;\quad v_n(x_n)\leqslant 0,\quad v_n(x_n)=0; \tag{1.7}$$
$$v_n(\pm 1)=u_n(\pm 1)=0$$

Here x is a coordinate of points in the contact domain, a and c are coordinates of the contact domain boundaries, $(x_k^\circ,\ y_k^\circ)$ are coordinates of the centre of the k-th crack, l_k and α_k are, respectively, half the crack length and the angle between the abscissa axes of the k-th local and the fundamental coordinate systems (the general form of the bodies making contact is displayed in Fig.1), x_k is a coordinate of points in the local coordinate system associated with the k-th crack, $q=q(x)$ and $p_k=p_k(x_k)$ are, respectively, the contact pressure and stress acting on the edge of the k-th crack, $v_k=v_k(x_k)$ and $u_k=u_k(x_k)$ are the jumps, respectively, in the normal and tangential displacements of the edges of the k-th crack, $f(x)$ is the shape of the base of the stamp, δ° is the proximity of the bodies, P is the force acting on the stamp in the normal direction, q_0 and b_0 are, respectively, the characteristic pressure and the half-width of the contact domain, $q_0 b_0=2\pi^{-1}P$, and $E'=E/(1-\nu^2)$ is the reduced elastic modulus.

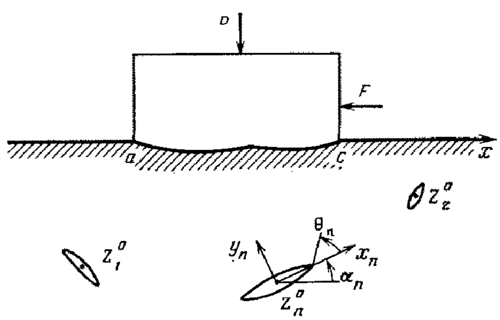

Fig.1

Therefore, for the given constants $z_k^\circ,\ \alpha_k,\ \delta_k\ (k=1,2,\ldots,N)$ and the function $f(x)$ from the equations and inequalities (1.2)-(1.7) it is required to determine the constant δ° and the functions $q(x),\ v_k(x_k),\ u_k(x_k)$ and $p_k(x_k)\ (k=1,2,\ldots,N)$.

2. Asymptotic investigation of the problem. We examine the case when all the cracks are small compared with the size of the contact area, i.e., $\delta_0=\max\delta_k\ll 1$. The examination of that structure of the system of cracks in an elastic half-plane for which the spacing between any two cracks substantially exceeds their size, i.e.,

$$z_n^\circ-z_k^\circ\gg\delta_0,\quad\forall n,\ k,\ n\neq k \tag{2.1}$$

is here of greatest practical interest. (The asymptotic relationship $g\sim h$ means that $(g\bar g)^{1/2}\sim(h,\bar h)^{1/2}$. It is evident that if $g\sim h$, then $\bar g\sim\bar h$. The asymptotic relationships $g\gg h$ and $g\ll h$ are determined analogously.)

Since the system of subsurface cracks belongs to the half-plane, then $\operatorname{Im} z_n^\circ \cdot \operatorname{Im} \bar{z}_k^\circ < 0$, $\forall n, k$. Consequently, it follows from (2.1) that

$$z_n^\circ - \bar{z}_k^\circ \gg \delta_0, \ \forall n, k, \ n \neq k \tag{2.2}$$

We moreover assume that the cracks in the elastic half-plane lie at depths under the surface that considerably exceed their size. This assumption can be expressed in the form

$$z_n^\circ - \bar{z}_n^\circ \gg \delta_0, \ \forall n \tag{2.3}$$

The following estimates obviously result from (2.1)-(2.3)

$$T_k - X_n \gg \delta_0, \ \bar{T}_k - X_n \gg \delta_0, \ \forall n, k, \ n \neq k \tag{2.4}$$
$$\bar{T}_k - T_k \gg \delta_0, \ x - T_k \gg \delta_0, \ \forall k, x$$

We now solve (1.2) for $q(x)$ by assuming that the shape of the stamp base $f(x)$ is such that $q(x)$ has a square root singularity at the points $x = a$ and $x = c$. By using /5/, we obtain

$$q(x) = q^\circ(x) - \frac{1}{\pi^2 \sqrt{(x-a)(c-x)}} \int_{-1}^{1} \frac{\sqrt{(t-a)(c-t)}\, dt}{t-x} \times \tag{2.5}$$
$$\sum_{k=1}^{N} \delta_k \int_{-1}^{1} \left\{ v_k'(\tau) \frac{\partial W_k^{\,r}(\tau, t)}{\partial t} - u_k'(\tau) \frac{\partial W_k^{\,i}(\tau, t)}{\partial t} \right\} d\tau$$

$$q^\circ(x) = \frac{1}{2\sqrt{(x-a)(c-x)}} \left\{ 1 + \frac{1}{\pi} \int_{-1}^{1} \frac{f'(t) \sqrt{(t-a)(c-t)}\, dt}{t-x} \right\} \tag{2.6}$$

We convert (2.5), for which we invert the orders of summation and integration. As a result of these transformations, from (2.5) and the integral differentiated once with respect to $\bar{T}_k$ (see /2, p.63/ and the relationships (1.3))

$$\int_{-b}^{b} \frac{\sqrt{b^2 - y^2}\, dy}{(y-x)(y-\bar{T}_k)} = -\pi \left(1 + \frac{\sqrt{\bar{T}_k^{\,2} - b^2}}{x - \bar{T}_k} \right)$$

the final form of the equation for $q(x)$ results

$$q(x) = q^\circ(x) - \frac{1}{\pi \sqrt{(x-a)(c-x)}} \sum_{k=1}^{N} \delta_k \int_{-1}^{1} [v_k'(t) Z_k^{\,r}(t, x) - \tag{2.7}$$
$$u_k'(t) Z_k^{\,i}(t, x)]\, dt$$

$$Z_k(t, x) = \frac{1}{\pi} \int_{a}^{c} \frac{\sqrt{(\tau-a)(c-\tau)}}{\tau - x} \frac{\partial W_k(t, \tau)}{\partial \tau} d\tau = \tag{2.8}$$
$$ie^{-i\alpha_k} \frac{(\bar{T}_k - T_k)\,[(c+a)(x+\bar{T}_k)/2 - ac - x\bar{T}_k]}{(x-\bar{T}_k)^2 \sqrt{(\bar{T}_k - a)(\bar{T}_k - c)}}$$

It follows from (2.8) and the estimates (2.1)-(2.4) that the kernels $Z_k(t, x)$, $D_n(t, x_n)$, $U_{nk}(t, x_n)$ and $V_{nk}(t, x_n)$ can be represented in the form of asymptotic series in powers of δ_k and δ_n (see (1.6)) that are regular for all x, x_n and t:

$$Z_k(t, x) = \sum_{j=0}^{\infty} (\delta_k t)^j Z_{kj}(x), \quad D_n(t, x_n) = \sum_{j=0}^{\infty} (\delta_n x_n)^j D_{nj}(t) \tag{2.9}$$
$$\{U_{nk}(t, x_n), V_{nk}(t, x_n)\} = \sum_{\substack{j+m=0 \\ j,\, m \geqslant 0}}^{\infty} (\delta_n x_n)^j (\delta_k t)^m \{U_{nkjm}, V_{nkjm}\}$$

We note that the quantities U_{nkjm} and V_{nkjm} are independent of $\delta_n, \delta_k, x_n, t$ since they are functions of the constants α_n, α_k, x_n°, y_n°, x_k° and y_k°. An analogous dependence also holds for the quantities $Z_{kj}(x)$ and $D_{kj}(t)$.

We will transfer directly to the asmyptotic solution of the system of equations and inequalities (2.6)-(2.8), (1.4)-(1.7) for $\delta_0 \ll 1$. We will seek the solution of the above-mentioned system by the method of regular perturbations /1/ in the form of asymptotic series in powers of δ_0

$$\{q, \delta^\circ, v_n, u_n, p_n\} = \sum_{j=0}^{\infty} \delta_0^{\,j} \{q_j, \delta_j^\circ, v_{nj}, u_{nj}, p_{nj}\} \tag{2.10}$$

We make an asymptotic analysis of (1.4) and (2.7). Substituting the representation

(2.10) into (1.4), we find by using the expansion (2.9) for the successive terms of the expansions p_n and v_n, u_n

$$Fv_{n0}'(x_n) = \pi p_{n0}(x_n) - \pi c_{n00}^r, \quad Fu_{n0}'(x_n) = -\pi c_{n00}^i \tag{2.11}$$

$$Fv_{n1}'(x_n) = \pi p_{n1}(x_n) - \pi c_{n01}^r \frac{\delta_n}{\delta_0} x_n - \pi c_{n10}^r \tag{2.12}$$

$$Fu_{n1}'(x_n) = -\pi c_{n01}^i \frac{\delta_n}{\delta_0} x_n - \pi c_{n10}^i$$

$$Fv_{n2}'(x_n) - \sum_{k=1}^{N} \left(\frac{\delta_k}{\delta_0}\right)^2 (\lambda_k U_{nk01}^r - \mu_k V_{nk01}^r) = \tag{2.13}$$

$$\pi p_{n2}(x_n) - \pi c_{n02}^r \left(\frac{\delta_n}{\delta_0} x_n\right)^2 - \pi c_{n11}^r \frac{\delta_n}{\delta_0} x_n - \pi c_{n20}^r$$

$$Fu_{n2}'(x_n) - \sum_{k=1}^{N} \left(\frac{\delta_k}{\delta_0}\right)^2 (\lambda_k U_{nk01}^i - \mu_k V_{nk01}^i) =$$

$$-\pi c_{n02}^i \left(\frac{\delta_n}{\delta_0} x_n\right)^2 - \pi c_{n11}^i \frac{\delta_n}{\delta_0} x_n - \pi c_{n20}^i, \ldots,$$

$$Fw(x) = \int_{-1}^{1} \frac{w(t)\,dt}{t - x}$$

$$c_{nkj}^r + i c_{nkj}^i = \frac{1}{\pi} \int_a^c q_k(\tau) \overline{D_{nj}(\tau)}\, d\tau \tag{2.14}$$

The quantities c_{nkj}^r and c_{nkj}^i are obviously independent of x_n. In deriving (2.11)-(2.13), we also took account of the relationships $v_{nk}(\pm 1) = u_{nk}(\pm 1) = 0$ and

$$(\lambda_k, \mu_k) = -\int_{-1}^{1} t\,(v_{k0}'(t), u_{k0}'(t))\, dt = \int_{-1}^{1} (v_{k0}(t), u_{k0}(t))\, dt \tag{2.15}$$

Furthermore, substituting (2.9) and (2.10) into (2.7), we obtain for the first terms of the asymptotic expansions of $q(x)$

$$q_0(x) = q^\circ(x), \; q_1(x) = 0 \tag{2.16}$$

$$q_2(x) = \frac{1}{\pi \sqrt{(x-a)(c-x)}} \sum_{k=1}^{N} \left(\frac{\delta_k}{\delta_0}\right)^2 [\lambda_k Z_{k1}^r(x) - \mu_k Z_{k1}^i(x)], \ldots \tag{2.17}$$

We turn now to a further analysis of (2.11)-(2.14). To do this, we perform an asymptotic analysis of the system of alternative equalities and inequalities (1.7) for $\delta_0 \ll 1$, where the functions v_n and p_n satisfy (2.10). We will have

$$\sum_{j=0}^{\infty} \delta_0^j p_{nj}(x_n) = 0, \quad \sum_{j=0}^{\infty} \delta_0^j v_{nj}(x_n) > 0; \tag{2.18}$$

$$\sum_{j=0}^{\infty} \delta_0^j p_{nj}(x_n) \leqslant 0, \quad \sum_{j=0}^{\infty} \delta_0^j v_{nj}(x_n) = 0$$

We assume that $v_{n0}(x_n) > 0$. Then from the first condition in (2.18) it follows that $p_{nj}(x_n) = 0, \forall j \geqslant 0$ for $\delta_0 \ll 1$, and the sign of $v_{nj}(x_n)$ does not influence the compliance with the second inequality in (2.18) for $j \geqslant 1$. We assume the opposite, i.e., $v_{n0}(x_n) = 0$. Then the realization of one of two cases is possible: a) $p_{n0}(x_n) < 0$; b) $p_{n0}(x_n) = 0$. In case a) we will have $p_n(x_n) < 0$ for $\delta_0 \ll 1$ outside the dependence on the values of $p_{nj}(x_n)$ for $j \geqslant 1$. We find $v_{nj}(x_n) = 0, \forall j \geqslant 0$ here from the last of the relationships (2.18). In case b) we obtain that $p_{n0}(x_n) = 0$ and $v_{n0}(x_n) = 0$ and the selection of the relationships for $p_{nj}(x_n)$ and $v_{nj}(x_n)$ for $j \geqslant 1$ is carried over the next approximation in $\delta_0 \ll 1$, which is done completely analogously.

From the last relationship in (1.7) and (2.10) it obviously follows that

$$v_{nk}(\pm 1) = u_{nk}(\pm 1) = 0, \; \forall n, \; k \tag{2.19}$$

1°. We consider the problem of the state of stress and strain of the material near cracks in the zeroth approximation. The problem is here described by (2.11) in combination with the system of equalities and inequalities (2.19) and

$$p_{n0}(x_n) = 0, \; v_{n0}(x_n) > 0; \quad p_{n0}(x_n) \leqslant 0, \quad v_{n0}(x_n) = 0 \tag{2.20}$$

which follows from an asymptotic analysis of (1.7) by using (2.10) (see the analysis of system

(2.18)).

We assume that $v_{n0}(x_n) > 0$, $\forall x_n \in (-1, 1)$. Then we obtain from (2.20) that $p_{n0}(x_n) = 0$, $\forall x_n \in (-1, 1)$, and we find $v_{n0}(x_n) = c^r_{n00}\sqrt{1-x_n^2} > 0$. Therefore we have $c^r_{n00} > 0$. It is seen that for $c^r_{n00} \leqslant 0$ the functions $v_{n0}(x_n) = 0$, $p_{n0}(x_n) = c^r_{n00} \leqslant 0$, $\forall x_n \in (-1, 1)$ satisfy relations (2.11) and (2.20). We obtain $u_{n0}(x_n) = c^i_{n0}\sqrt{1-x_n^2}$ /5/ from the second equation of (2.11) and (2.19). Therefore, the solution of problem (2.11), (2.19), (2.20) has the form ($\theta(\cdot)$ is the Heaviside function)

$$v_{n0}(x_n) = c^r_{n00}\theta(c^r_{n00})\sqrt{1-x_n^2}, \quad u_{n0}(x_n) = c^i_{n00}\sqrt{1-x_n^2}, \qquad (2.21)$$
$$p_{n0}(x_n) = c^r_{n00}\theta(-c^r_{n00})$$

It follows from an analysis of the system of alternative equalities and inequalities (2.18) that $v_{n0}(x_n) > 0$ and $p_{nj}(x_n) = 0$, $x_n \in (-1, 1)$, $\forall j \geqslant 0$ for $c^r_{n00} > 0$ (see (2.21)) and the sign of $v_{nj}(x_n)$, $\forall j \geqslant 1$ is not essential. Consequently, we obtain from (2.12) /5/ by using (2.16), (2.19) and (2.20) for $c^r_{n00} > 0$

$$v_{n1}(x_n) = \frac{\delta_n}{2\delta_0}c^r_{n01}x_n\sqrt{1-x_n^2}, \quad u_{n1}(x_n) = \frac{\delta_n}{2\delta_0}c^i_{n01}x_n\sqrt{1-x_n^2} \qquad (2.22)$$

We find analogously /5/ from (2.13), (2.19) and (2.20) for $c^r_{n00} > 0$

$$v_{n2}(x_n) = \left[A_n^r + \frac{1}{6}B_n^r(1+2x_n^2)\right]\sqrt{1-x_n^2} \qquad (2.23)$$
$$u_{n2}(x_n) = \left[A_n^i + \frac{1}{6}B_n^i(1+2x_n^2)\right]\sqrt{1-x_n^2}$$
$$A_n^r = c^r_{n20} - \frac{1}{\pi}\sum_{k=1}^{N}\left(\frac{\delta_k}{\delta_0}\right)^2(\lambda_k U^r_{nk01} - \mu_k V^r_{nk01}) \qquad (2.24)$$
$$B_n^r = \left(\frac{\delta_n}{\delta_0}\right)^2 c^r_{n02}$$

To determine the constants A_n^i and B_n^i it is sufficient to replace the superscript r by i in (2.24).

For $c^r_{n00} < 0$ it follows from an analysis of the system of alternative equalities and inequalities (2.18) that (see (2.21)) $p_{n0}(x_n) < 0$ and $v_{nj}(x_n) = 0$, $x_n \in (-1, 1)$, $\forall j \geqslant 0$, and the sign of $p_{nj}(x_n)$ is not essential for $j \geqslant 1$. Consequently, we obtain for $c^r_{n00} < 0$ from (2.12), (2.13), (2.15)-(2.17), (2.19) and (2.20)

$$p_{n1}(x_n) = \delta_0^{-1}\delta_n c^r_{n01}x_n \qquad (2.25)$$
$$p_{n2}(x_n) = A_n^r + B_n^r x_n^2 \qquad (2.26)$$

while the function $u_{n1}(x_n)$ is determined from (2.22).

2°. We examine the case $c^r_{n00} = 0$ when we have $v_{n0}(x_n) = p_{n0}(x_n) = 0$, $\forall x_n \in (-1, 1)$ in the zeroth approximation (see (2.21)). The summation in the system of equalities and inequalities (2.18) here starts with $j = 1$. Hence, it is necessary to analyse system (2.12)-(2.17) in combination with the equalities and inequalities (2.18). For $c^r_{n00} = 0$ an analogue of the relationship (2.20) follows from (2.18)

$$p_{n1}(x_n) = 0, \ v_{n1}(x_n) > 0; \ p_{n1}(x_n) \leqslant 0, \ v_{n1}(x_n) = 0 \qquad (2.27)$$

We will assume that $c^r_{n01} > 0$. Then from the form of the expressions for $v_{n1}(x_n)$ in (2.23), which is obtained under the condition $p_{nj}(x_n) = 0$, $x_n \in (-1, 1)$, it can be assumed that the segment $(-1, 1)$ occupied by the crack is separated into the segments $(-1, b_{n1})$ and $(b_{n1}, 1)$ on which the relationships $v_n(x_n) = 0$ and $v_n(x_n) > 0$ are satisfied. For convenience we introduce into (1.4)-(1.6) and (2.9) the following change of variables

$$x_n = [1 + b_{n1} + (1 - b_{n1})y]/2, \ t = [1 + b_{n1} + (1 - b_{n1})\tau]/2$$

The constant b_{n1} is unknown here and can also be expanded in powers of δ_0

$$b_{n1} = \sum_{j=0}^{\infty}\beta_{nj}\delta_0^j \qquad (2.28)$$

As a result of the above-mentioned transformations, by using (2.19), (2.27), we obtain from (1.4)-(1.6) and (2.9) for $v_{n1}(x_n)$, $x_n \in (\beta_{n0}, 1)$

$$\int_{-1}^{1}\frac{v'_{n1}(\tau)\,d\tau}{\tau - y} = -\pi\frac{\delta_n}{\delta_0}c^r_{n01}\frac{1-\beta_{n0}}{4}[1 + \beta_{n0} + (1-\beta_{n0})y], \quad v_{n1}(\pm 1) = 0 \qquad (2.29)$$

We note that the constant β_{n0} is unknown in (2.29) and is determined from the condition

$$p_{n1}(-1) = 0 \tag{2.30}$$

Then we obtain from (2.29) /5/

$$v_{n1}(y) = \frac{\delta_n}{\delta_0} c_{n01}^r \frac{1-\beta_{n0}}{4}\left(1+\beta_{n0}+\frac{1-\beta_{n0}}{2}y\right)\sqrt{1-y^2}\,\theta(1-y^2) \tag{2.31}$$

We use (1.4) transformed by the above-mentioned method and the integral /6/

$$\int_{-1}^{1}\frac{d\tau}{\sqrt{1-\tau^2}(\tau-y)} = \begin{cases} 0, & |y|\leqslant 1 \\ -\dfrac{\pi}{\sqrt{y^2-1}}\operatorname{sign} y, & |y|>1 \end{cases} \tag{2.32}$$

to determine the function p_{n1}.

As a result of the transformations described we obtain

$$p_{n1}(y) = -\frac{\delta_n}{2\delta_0} c_{n01}^r \frac{\operatorname{sign} y}{\sqrt{y^2-1}}\left[-(1+\beta_{n0})y + (1-\beta_{n0})\left(\frac{1}{2} - y^2\right)\right]\theta(y^2-1) \tag{2.33}$$

We now determine $\beta_{n0} = -{}^1/_3$. by using (2.30) and (2.33). Substituting the value found for β_{n0} into (2.31) and (2.33), and returning to the variable x_n $(x_n = [1+\beta_{n0}+(1-\beta_{n0})y]/2)$, we will have for $c_{n01}^r > 0$

$$v_{n1}(x_n) = \frac{\sqrt{3}\,\delta_n}{18\delta_0} c_{n01}^r (3x_n+1)\sqrt{1+2x_n-3x_n^2}\,\theta(1+2x_n-3x_n^2) \tag{2.34}$$
$$p_{n1}(x_n) = \frac{\sqrt{3}\,\delta_n}{9\delta_0} c_{n01}^r (3x_n-2)\sqrt{\frac{3x_n+1}{x_n-1}}\,\theta(3x_n^2-2x_n-1)$$

Analogously for $c_{n00}^r = 0$ and $c_{n01}^r < 0$ we obtain $\beta_{n0} = {}^1/_3$ and expressions (2.34) for v_{n1} and p_{n1} in which the signs of x_n and the right sides of the equalities are changed.

We note that for $c_{n01}^r > 0$ and $c_{n01}^r < 0$ the functions u_{n1} and u_{n2} are determined, respectively, from the relationships (2.22)-(2.24). We will now obtain the function v_{n2} for the cases under consideration. It can be shown that for $c_{n00}^r = 0$ and $c_{n01}^r > 0$ the function $v_{n2}(y)$ satisfies the problem

$$Fv_{n2}'(y) = -\pi\frac{1-\beta_{n0}}{2}\left\{\frac{1}{4}B_n^r[1+\beta_{n0}+(1-\beta_{n0})y]^2 + A_n^r\right\} + \pi\frac{\delta_n}{\delta_0}c_{n01}^r\frac{1-\beta_{n0}}{2}\left(y+\frac{\beta_{n0}}{1-\beta_{n0}}\right)\beta_{n1}, \quad v_{n2}(\pm 1) = 0 \tag{2.35}$$

It is here necessary to note that the sign of the function $v_{n2}(y)$ does not influence the satisfaction of the first inequality in (2.18) for $|y|\leqslant 1$, while the constant β_{n1} is unknown and determined from the solution of the equation

$$p_{n2}(-1) = 0 \tag{2.36}$$

The solution of (2.35) for $\beta_{n0} = -{}^1/_3$ has the form /5/

$$v_{n2}(y) = \frac{2}{3}\left[A_n^r + \frac{1}{27}B_n^r(5+6y+4y^2) + \frac{1}{4}\frac{\delta_n}{\delta_0}c_{n01}^r\beta_{n1}(1-2y)\right]\sqrt{1-y^2} \tag{2.37}$$

As a result of a transformation of the appropriate equations we obtain for $p_{n2}(y)$

$$p_{n2}(y) = \frac{1}{2}\frac{\delta_n}{\delta_0}c_{n01}^r\beta_{n1}(1-y) + \frac{1}{4}B_n^r[1+\beta_{n0}+(1-\beta_{n0})y]^2 + A_n^r + \frac{2}{\pi(1-\beta_{n0})^2}[\beta_{n1}Fv_{n1}'(y) + (1-\beta_{n0})Fv_{n2}'(y)] \tag{2.38}$$

Furthermore, by using (2.31)-(2.33) and (2.37) for $\beta_{n0} = -{}^1/_3$ we find from (2.36)

$$\beta_{n1} = -\frac{4}{3}\left(A_n^r + \frac{1}{9}B_n^r\right)\Big/\left(c_{n01}^r\frac{\delta_n}{\delta_0}\right) \tag{2.39}$$

Finally, by using (2.39), we obtain from (2.37) and (2.38)

$$v_{n2}(\sigma_n) = \frac{\sqrt{3}}{9}\left[A_n{}^r + \frac{1}{9}B_n{}^r(3\sigma_n+1)\right](3\sigma_n+1)\times \sqrt{1+2\sigma_n-3\sigma_n{}^2}\,\theta(1+2\sigma_n-3\sigma_n{}^2) \tag{2.40}$$

$$p_{n2}(\sigma_n) = \frac{\sqrt{3}}{9}\left[A_n{}^r(3\sigma_n-2) + \frac{1}{9}B_n{}^r(-5-15\sigma_n+27\sigma_n{}^2)\right]\times \sqrt{\frac{3\sigma_n+1}{\sigma_n-1}}\,\theta(3\sigma_n{}^2-2\sigma_n-1)$$

$$\sigma_n = x_n \operatorname{sign} c_{n01}^r$$

Analogously, for $c_{n00}^r = 0$ and $c_{n01}^r < 0$ we obtain (2.39) and (2.40) for v_{n2} and p_{n2}. We evidently have from (2.12), (2.21) and (2.27) for $c_{n00}^r = c_{n01}^r = 0$

$$v_{nj}(x_n) = p_{nj}(x_n) = 0, \ \forall x_n \in (-1,\ 1) \ (j = 0,1) \tag{2.41}$$

and the functions $u_{n1}(x_n)$ and $u_{n2}(x_n)$ are also determined from the relationships (2.22)-(2.24). The function $v_{n2}(x_n)$ is here not determined by the equality from (2.23).

3°. We consider the case $c_{n00}^r = c_{n01}^r = 0$ when the equalities (2.41) are satisfied. The summation in (2.18) here starts with $j = 2$ and the analogue of the relationships (2.20) is valid for $v_{n2}(x_n)$ and $p_{n2}(x_n)$

$$p_{n2}(x_n) = 0,\ v_{n2}(x_n) > 0;\ p_{n2}(x_n) \leqslant 0,\ v_{n2}(x_n) = 0 \tag{2.42}$$

Furthermore, it will be shown that depending on the value of the ratio $A_n{}^r/B_n{}^r$ each crack can have a different configuration: one or two symmetrically arranged sections with joined edges. The evenness of the functions $v_{n2}(x_n)$, $u_{n2}(x_n)$ and $p_{n2}(x_n)$ follows from (2.13), (2.15) and conditions (2.19).

We first consider the case when $v_{n2}(x_n) = 0, \forall x_n \in (-1, 1)$. Then we obtain from (2.13), (2.15) and (2.24)

$$v_{n2}(x_n) = 0,\ p_{n2}(x_n) = A_n{}^r + B_n{}^r x_n{}^2 \tag{2.43}$$

where the functions $v_{n2}(x_n)$ and $p_{n2}(x_n)$ from (2.43) satisfy (2.42) for

$$A_n{}^r \leqslant 0,\ B_n{}^r \leqslant 0 \ \text{ or } \ A_n{}^r + B_n{}^r \leqslant 0,\ B_n{}^r > 0 \tag{2.44}$$

We consider the case when there is one symmetrically located section with open edges in the segment $(-1, 1)$ occupied by the n-th crack. Let this be the segment $(-b_{n2}, b_{n2})$. The constant b_{n2} here is not arbitrary but satisfies the relations: a) $p_{n2}(b_{n2}) = 0,\ b_{n2} < 1$ (see (2.42)); or b) $b_{n2} = 1$. Moreover, it follows from (2.24) that $v_{n2}(x_n) > 0,\ x_n \in (-b_{n2}, b_{n2})$ and $p_{n2}(x_n) \leqslant 0,\ x_n \in (-1, -b_{n2}) \cup (b_{n2}, 1)$ in case a).

Taking account of the conditions $v_{n2}(\pm b_{n2}) = 0$ in case a), the solution of the first equation in (2.13), (2.4) has the form /5/

$$v_{n2}(x_n) = \frac{1}{2}\left[2A_n{}^r + {}^1/_3 B_n{}^r(b_{n2}^2 + 2x_n{}^2)\right]\times \sqrt{b_{n2}^2 - x_n{}^2}\,\theta(b_{n2}^2 - x_n{}^2) \tag{2.45}$$

$$p_{n2}(x_n) = |x_n|\left\{\left[A_n{}^r + B_n{}^r(x_n{}^2 - {}^1/_2 b_{n2}^2)\right]\Big/\sqrt{x_n{}^2 - b_{n2}^2}\right\}\times \theta(x_n{}^2 - b_{n2}^2)$$

Here we have used the fact that $p_{n2}(x_n) = 0,\ x_n \in (-b_{n2}, b_{n2})$.

We now find from the condition $p_{n2}(b_{n2}) = 0$

$$b_{n2}{}^2 = -2A_n{}^r/B_n{}^r \tag{2.46}$$

Substituting the expression for b_{n2} from (2.46) into (2.45), we obtain

$$v_{n2}(x_n) = -\frac{B_n{}^r}{3}\left(-\frac{2A_n{}^r}{B_n{}^r} - x_n{}^2\right)^{3/2}\theta\left(-\frac{2A_n{}^r}{B_n{}^r} - x_n{}^2\right) \tag{2.47}$$

$$p_{n2}(x_n) = B_n{}^r|x_n|\left(\frac{2A_n{}^r}{B_n{}^r} + x_n{}^2\right)^{1/2}\theta\left(\frac{2A_n{}^r}{B_n{}^r} + x_n{}^2\right)$$

Furthermore, from (2.46) and (2.47) and the conditions listed above (corollaries of (2.42)) and the condition $b_{n2}{}^2 \geqslant 0$ we obtain relations in the constants $A_n{}^r$ and $B_n{}^r$

$$A_n{}^r > 0,\ B_n{}^r < 0,\ 2A_n{}^r + B_n{}^r < 0 \tag{2.48}$$

Let us study case b). It is seen that the solution (2.45) in which we must put $b_{n2} = 1$, is valid. Consequently, we find from (2.45)

$$v_{n2}(x_n) = [A_n{}^r + {}^1/_6\, B_n{}^r (1 + 2x_n{}^2)]\sqrt{1 - x_n{}^2} \tag{2.49}$$

It follows from system (2.42) that $v_{n2}(x_n) > 0$, $x_n \in (-1, 1)$, where this last inequality is valid when one of the following system of inequalities holds:

$$A_n{}^r > 0,\quad B_n{}^r < 0,\quad 2A_n{}^r + B_n{}^r \geqslant 0,\quad \text{or}\quad B_n{}^r > 0,$$
$$6A_n{}^r + B_n{}^r \geqslant 0,\quad \text{or}\quad A_n{}^r > 0,\quad B_n{}^r = 0. \tag{2.50}$$

It follows from (2.44), (2.48) and (2.50) that only the domain of the parameters $A_n{}^r$ and $B_n{}^r$ subjected to the inequalities

$$A_n{}^r < 0,\quad B_n{}^r > 0,\quad A_n{}^r + B_n{}^r > 0,\quad 6A_n{}^r + B_n{}^r \leqslant 0 \tag{2.51}$$

remains uninvestigated.

We will now investigate this domain of the parameters. We assume that there are two symmetrically arranged zones $(-b_{n2}, -a_{n2})$ and (a_{n2}, b_{n2}) in the segment $(-1, 1)$ on which the edges of the n-th crack are open, i.e., $v_{n2}(x_n) > 0$. It can be shown that the constants $b_{n2} = 1$, a_{n2} from

$$\frac{E(\sqrt{1 - a_{n2}^2})}{K(\sqrt{1 - a_{n2}^2})} = a_{n2}^2\, \frac{3a_{n2}^2 - 1 + 6A_n{}^r/B_n{}^r}{a_{n2}^2 + 1 + 6A_n{}^r/B_n{}^r} \tag{2.52}$$

and the functions

$$v_{n2}(x_n) = \left\{\frac{\sqrt{(1 - x_n{}^2)(x_n{}^2 - a_{n2}^2)}}{|x_n|}\left[A_n{}' + \frac{1}{3}B_n{}^r\left(\frac{1 + a_{n2}^2}{2} + x_n{}^2\right)\right] - \right. \tag{2.53}$$
$$\left.\left(A_n{}^r + B_n{}^r\frac{1 + a_{n2}^2}{6}\right)\left[E(\varkappa, \mu) - \frac{E(\mu)}{K(\mu)}F(\varkappa, \mu)\right]\right\}\theta(x_n{}^2 - a_{n2}^2)$$
$$p_{n2}(x_n) = -\frac{\theta(a_{n2}^2 - x_n{}^2)}{\sqrt{(x_n{}^2 - 1)(x_n{}^2 - a_{n2}^2)}}\left\{\left(B_n{}^r\frac{1 + a_{n2}^2}{2} - A_n{}^r\right)\left[\frac{E(\mu)}{K(\mu)} - \right.\right.$$
$$\left.\left. x_n{}^2\right] - \frac{1}{3}B_n{}^r\left[2(1 + a_{n2}^2)\frac{E(\mu)}{K(\mu)} - a_{n2}^2 - 3x_n{}^4\right]\right\}$$
$$\varkappa = \arcsin\left[\frac{1}{x_n}\sqrt{\frac{x_n{}^2 - a_{n2}^2}{1 - a_{n2}^2}}\right],\quad \mu = \sqrt{1 - a_{n2}^2}$$

where $F(\varkappa, \eta)$ and $E(\varkappa, \eta)$ are elliptic, and $K(\eta)$ and $E(\eta)$ complete elliptic integrals, respectively, of the first and second kinds /7/, are a solution of the problem for the first equation from (2.13) with the conditions $v_{n2}(\pm b_{n2}) = v_{n2}(\pm a_{n2}) = 0$; $p_{n2}(x_n) \leqslant 0$ for $|x_n| < a_{n2}$ and $|x_n| > b_{n2}$. An expressions for the difference of the integrals

$$\left(\int_{-b}^{-a} - \int_{a}^{b}\right)\frac{dt}{\sqrt{(b^2 - t^2)(t^2 - a^2)}\,(t - x)} =$$
$$\frac{\pi\, \text{sign}\,(x^2 - a^2)}{\sqrt{(b^2 - x^2)(a^2 - x^2)}}\,\theta[(b^2 - x^2)(a^2 - x^2)]$$

obtained according to the theory of residues was used in deducing (2.53).

The solution of (2.52) for different values of $A_n{}^r/B_n{}^r$ is represented in Fig.2.

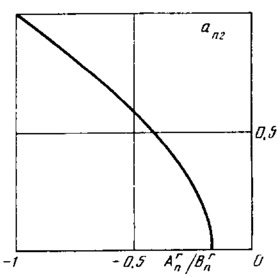

Fig.2

We note that the function $u_{n2}(x_n)$ in the cases studied in Sect. 3 is, as before, determined from (2.23) while the quantities λ_k and μ_k from (2.15) that are in (2.17), (2.23), (2.24) and (2.26) equal (see (2.21))

$$(\lambda_k,\ \mu_k) = {}^1/_2\pi\,(c_{k00}^r\,\theta(c_{k00}^r),\ c_{l\,00}^i) \tag{2.54}$$

4°. We now calculate the intensity factors $k_{1n}{}^\pm$ and $k_{2n}{}^\pm$ as well as the angles of the initial propagation directions of the n-th crack $\theta_n{}^\pm$. The superscript $\pm$ here refers, respectively, to the crack tips with $x_n = \pm 1$. Following /2, 3/, we will have

$$k_{1n}^\pm + ik_{2n}^\pm = \mp \lim_{x_n \to \pm l_n}\left\{\frac{E'}{4}\sqrt{\frac{l_n{}^2 - x_n{}^2}{l_n}}\,[v_n{}'(x_n) + iu_n{}'(x_n)]\right\} \tag{2.55}$$

Inserting the dimensionless quantities $k_{jn}{}^{\pm\prime} = k_{jn}{}^\pm/(q_0\sqrt{l_n})$ for $k_{jn}{}^\pm (j = 1, 2)$ we find from (1.1) and (2.55) (the primes are omitted)

$$k_{1n}^\pm + ik_{2n}^\pm = \mp \lim_{x_n \to \pm 1}\sqrt{1 - x_n{}^2}\,[v_n{}'(x_n) + iu_n{}'(x_n)] \tag{2.56}$$

By using the solutions obtained we evaluate the stress intensity factors

$$k_{2n}^{\pm} = c_{n00}^{i} \pm {}^1/_2\delta_n c_{n01}^{i} + \delta_0^2 c_{n20}^{i} + {}^1/_2\delta_n^2 c_{n02}^{i} - \Delta^i + \ldots \tag{2.57}$$

$$k_{1n}^{\pm} = c_{n00}^{r} \pm {}^1/_2\delta_n c_{n01}^{r} + \delta_0^2 c_{n20}^{r} + {}^1/_2\delta_n^2 c_{n02}^{r} - \Delta^r + \ldots, \tag{2.58}$$

$$c_{n00}^{r} > 0; \quad k_{1n}^{\pm} = 0, \quad c_{n00}^{r} < 0$$

$$k_{1n}^{\pm} = \frac{\sqrt{3}}{9} \delta_n c_{n01}^{r} [\pm 7 - 3\theta (c_{n01}^{r})] \left[\frac{1 \pm \theta (c_{n01}^{r})}{1 \pm 3\theta (c_{n01}^{r})}\right]^{1/2} + \frac{4\sqrt{6}}{9} \left(\delta_0^2 c_{n20}^{r} + \frac{4}{9} \delta_n^2 c_{n02}^{r} - \Delta^r\right) + \ldots, \quad c_{n01}^{r} \neq 0 \tag{2.59}$$

For $c_{n00}^{r} = c_{n01}^{r} = 0$ we obtain, respectively

$$k_{1n}^{\pm} = 0 \ \text{ for } A_n^r \leqslant 0, \quad B_n^r \leqslant 0; \quad A_n^r + B_n^r \leqslant 0, \quad B_n^r > 0 \tag{2.60}$$

$$A_n^r > 0, \quad B_n^r < 0, \quad 2A_n^r + B_n^r < 0;$$

$$k_{1n}^{\pm} = \delta_0^2 c_{n20}^{r} + {}^1/_2\delta_n^2 c_{n02}^{r} - \Delta^r + \ldots \ \text{ for } A_n^r > 0, \quad B_n^r < 0,$$

$$2A_n^r + B_n^r \geqslant 0; \quad B_n^r \geqslant 0, \quad 6A_n^r + B_n^r > 0$$

$$k_{1n}^{\pm} = \frac{1}{\sqrt{1 - a_{n2}^2}} \left\{\left[\delta_n^2 \frac{1 + a_{n2}^2}{2} c_{n02}^{r} - \delta_0^2 c_{n20}^{r} + \Delta^r\right] \times \left[\frac{E(\sqrt{1 - a_{n2}^2})}{K(\sqrt{1 - a_{n2}^2})} - 1\right] + \delta_n^2 c_{n02}^{r} \left[1 - \frac{1}{3}\left(2(1 + a_{n2}^2) \times \frac{E(\sqrt{1 - a_{n2}^2})}{K(\sqrt{1 - a_{n2}^2})} - a_{n2}^2\right)\right]\right\} + \ldots$$

$$A_n^r < 0, \quad B_n^r > 0, \quad A_n^r + B_n^r > 0, \quad 6A_n^r + B_n^r \leqslant 0$$

$$\Delta^r = \frac{1}{2} \sum_{k=1}^{N} \delta_k^2 [c_{k00}^{r} \theta (c_{k00}^{r}) U_{nk01}^{r} - c_{k00}^{i} V_{nk01}^{r}] \tag{2.61}$$

The constant Δ^i is determined from (2.61) by replacing U_{nk01}^{r} (V_{nk01}^{r}) and U_{nk01}^{i} (V_{nk01}^{i}).

After having computed the intensity factors $k_{jn}^{\pm}$ by using one of the formulas (2.57)-(2.60), following /2/, we determine the angles

$$\theta_n^{\pm} = 2 \operatorname{arctg} \frac{k_{1n}^{\pm} - \sqrt{(k_{1n}^{\pm})^2 + 8(k_{2n}^{\pm})^2}}{4k_{2n}^{\pm}} \tag{2.62}$$

3. Qualitative and numerical results. We will not a number of qualitative properties of the solution of the problem. It follows from (2.10), (2.14)-(2.17) and (2.54) that the contact pressure $q(x)$ experiences the influence of the cracks, except starting with the second approximation, i.e., the magnitude of this influence is proportional to δ_0^2. The zeroth terms of the expansions of the jumps in the displacements $v_n(x_n)$ and $u_n(x_n)$ and the stress $p_n(x_n)$ (see (2.10) and (2.21)) on the edges of the crack under consideration are determined by the mutual location of the cracks and the contact domain as well as the pressure distribution and the crack orientation and are independent of its size and the presence of other cracks (see (2.14)). It follows from (2.55), (2.57) and (2.58) that the zeroth terms of the expansions of the dimensional intensity factors $k_{1n}^{\pm}$ and $k_{2n}^{\pm}$ are proportional to $\sqrt{l_n}$ and the properties in the rest are completely analogous to the properties of the zeroth terms of the expansions of v_n, u_n and p_n. The first terms of the expansions of $v_n, u_n, p_n, k_{1n}^{\pm}$ and $k_{2n}^{\pm}$ moreover depend on the crack size, while the second and subsequent terms also depend on the presence of other cracks.

By virtue of the dependence of the solution constructed on just the integrals c_{nkj}^{r} and c_{nkj}^{i} (see (2.14)), a number of generalizations is possible. Problems taking account of surface roughness /8/, sliding friction (linear and non-linear) /9/, friction with slippage and adhesion /10/, lubrication /4/, wear /11/, etc. can be investigated analogously. The zeroth term of the pressure asymptotic $q_0(x)$ will here be determined by the solution of the appropriate problem without taking friction into account, $q_1(x) = 0$, while the second term of the expansion will depend on the presence, location, size, and orientation of the cracks. When studying problems with previously unknown contact area, its boundaries will be independent of the presence of cracks to $O(\delta_0)$ accuracy. Expressions for the functions v_n, u_n, p_n and the constants $k_{1n}^{\pm}$ and $k_{2n}^{\pm}$ agree with the corresponding expressions obtained in Sect.2 (it should here by kept in mind that the role of the quantities c_{nkj}^{r} and c_{nkj}^{i} will here be played by different integrals in general). Distinctions occur only in the case of a problem with a previously unknown contact domain, in the second terms of the expansions of these functions and are associated with variations in the contact domain boundaries. Nevertheless, even in this case the solution of problems for v_{n2}, u_{n2} and p_{n2} can be reduced to a form analogous to that investigated in Sect.2.

We examine the results obtained for the problem with friction in detail in the case of total slippage. The first equation in (1.2) is here replaced by the equation /4, 9/

$$f(x) - \lambda \int_a^x \psi(q(t))\,dt + \frac{2}{\pi}\int_a^c q(t)\ln\frac{1}{|x-t|}\,dt - \tag{3.1}$$
$$\frac{2}{\pi}\sum_{k=1}^{N}\delta_k \int_{-1}^{1} \{v_k'(t)\,W_k^r(t,x) - u_k'(t)\,W_k^i(t,x)\}\,dt = \delta^\circ$$

and, in addition, the right-hand sides of (1.4) are modified (see (3.3)). In (3.1) $\psi = \psi(q)$ is a fairly smooth pressure function governing the friction stress, λ is a constant playing the part of the coefficient of friction. The solution of (3.1) for $\lambda \ll 1$ and $\delta_0 \ll 1$ can be constructed by methods from /9/. When examining a problem with a previously unknown contact domain the conditions $q(a) = q(c) = 0$ must be appended to (3.1). Moreover, performing the analysis described in Sect.2, we can see the validity of formulas (2.21)-(2.26), (2.34), (2.40), (2.43), (2.44), (2.46)-(2.54), (2.57)-(2.61) with the appropriate replacement of the expressions for $c_{nkj}^r + ic_{nkj}^i$, $q_k(x)$ from (2.6), (2.15)-(2.17).

In particular, by confining ourselves to a consideration of two-term expansions for the problem with previously unknown boundaries $a = -b$, $c = b$ we obtain for $\psi(q) = q$ and $f(x) = (x + d)^2$ /5/

$$q_0(x) = \cos\pi\gamma\,(b_0 + x)^{1/2-\gamma}(b_0 - x)^{1/2+\gamma}, \quad b_0 = (1-4\gamma^2)^{-1/2} \tag{3.2}$$
$$d_0 = -2\gamma b_0, \quad \gamma = \frac{1}{\pi}\operatorname{arctg}\frac{\lambda}{2}; \quad q_1(x) = 0$$

$$c_{nkj}^r + ic_{nkj}^i = \frac{1}{\pi}\int_{-b}^{b} q_k(t)\,[\overline{D_{nj}(t)} - \lambda\overline{G_{nj}(t)}]\,dt \tag{3.3}$$

In (3.3) we took $\tau(x) = -\lambda q(x)$ and the functions $G_{nj}(t)$ are coefficients of the expansions in powers of $\delta_n x_n$ for the kernel $G_n(t, x_n)$ that determines the influence of the tangential stress to the half-plane surface on the state of stress and strain of the elastic material near the cracks. The functions $G_n(t, x_n)$ and $G_{nj}(t)$ are determined from the relations /2/

$$G_n(t, x_n) = \frac{1}{2}\left[\frac{1}{t - X_n} + \frac{1 - e^{-2i\alpha_n}}{t - \bar{X}_n} - e^{-2i\alpha_n}\frac{t - X_n}{(t - \bar{X}_n)^2}\right] \tag{3.4}$$
$$G_n(t, x_n) = \sum_{j=0}^{\infty}(\delta_n x_n)^j\,G_{nj}(t)$$

We present below numerical results for k_{1n}^+ and k_{2n}^+ obtained on the basis of the formulas presented for $y_n^\circ = -0.2$ and $\delta_n = 0.1$, where the coefficient of friction $\lambda = 0.1$ corresponds to curve *1*, and $\lambda = 0.2$ to curve *2*. For $\lambda = 0.1$ we have $\gamma = 0.0159$, $b_0 = 1.0005$, $d_0 = -0.0318$, and for $\lambda = 0.2 - \gamma = 0.0317$, $b_0 = 1.002$, $d_0 = -0.0635$.

Curves of the dependence of the stress intensity factor k_{1n}^+ for the cases of horizontal cracks $\alpha_n = 0$ (Fig.3) and vertical cracks $\alpha_n = \pi/2$ (Fig.4) behave analogously. At the same time, for $\alpha_n = \pi/2$ the value of k_{1n}^+ is more than an order greater than for the case $\alpha_n = 0$. In addition, the quantity k_{1n}^+ increases substantially as the coefficient of friction increases, and reaches a maximum in the immediate vicinity of the contact domain boundary on the side opposite to the direction of stamp motion. When there is no friction $(\lambda = 0)$ the stresses are compressive everywhere in the half-plane, the cracks are closed, and consequently, the intensity factor k_{1n}^+ is identically zero. As the degree of crack submersion y_n° increases, the intensity coefficient k_{1n}^+ (for those x_n° for which $k_{1n}^+ > 0$) for the vertical crack $(\alpha_n = \pi/2)$ decreases monotonically while for the horizontal crack $(\alpha_n = 0)$ it first grows to the maximum value and then decreases to zero. In both cases k_{1n}^+ vanishes for $x_n^\circ \sim 1$ for $|y_n^\circ| \sim 1$.

The shear stress intensity factor k_{2n}^+ behaves differently. In both cases, for $\alpha_n = 0$ (Fig.5) and $\alpha_n = \pi/2$ (Fig.6) the curves $k_{2n}^+ = k_{2n}^+(x_n^\circ)$ reach extremal values in the immediate neighbourhood of the contact domain boundaries. Moreover, the behaviour of the quantities k_{2n}^+ as a function of y_n° is different for different x_n°. For instance, for $x_n^\circ = -5, -2, -1, 1$ the functions $k_{2n}^+(y_n^\circ)$ have one extremum in the cases of horizontal and vertical cracks; while for $x_n^\circ = 2.5$ for a vertical crack and $x_n^\circ = 5$ for a horizontal crack the functions $k_{2n}^+(y_n^\circ)$ have two extrema. The crack orientation obviously exerts a strong influence on the quantity k_{2n}^+ and quite a small influence on the value of the coefficient of friction λ.

We note that the quantities k_{1n}^+ and k_{1n}^- (k_{2n}^+ and k_{2n}^-) are close for $\delta_0 \ll 1$.

In conclusion, it must be emphasized that the results obtained indicate the strong influence of friction on the process of fracture of elastic bodies and can be used to estimate the longevity (margin) on the basis of a kinetic equation of crack development of the Paris type. Moreover, we note that the analysis of the problem presented explains completely the fatigue test results of a model roller bearing /12, p.39/ for a different friction stress level.

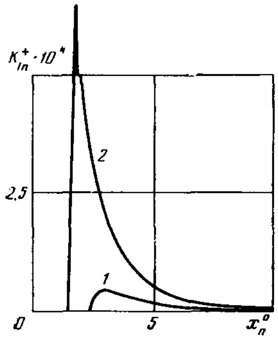

Fig.3

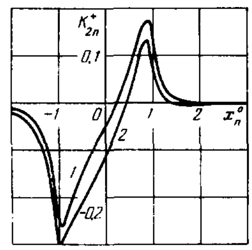

Fig.4

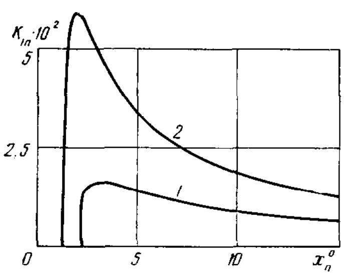

Fig.5

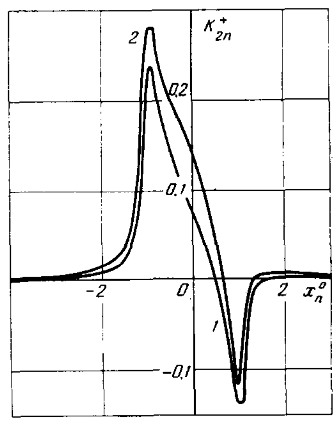

Fig.6

REFERENCES

1. VAN DYKE M., Perturbation Methods in Fluid Mechanics /Russian translation/, Mir, Moscow, 1967.
2. PANASYUK V.V., SAVRUK M.P. and DATSYSHIN A.P., Stress Distribution around Cracks in Plates and Shells. Naukova Dumka, Kiev, 1976.
3. SAVRUK M.P., Two-dimensional Elasticity Problems for Bodies with Friction. Naukova Dumka, Kiev, 1981.
4. KUDISH I.I., The contact-hydrodynamic problem of lubrication theory for elastic bodies with cracks, PMM, 48, 5, 1984.
5. SHTAERMAN I.YA., The Contact Problem of Elasticity Theory. Gostekhizdat, Moscow-Leningrad, 1949.
6. PYKHTEYEV G.N., Exact Methods for Evaluating Cauchy-Type Integrals. Nauka, N vosibirsk, 1980.
7. GRADSHTEIN I.S. and RYZHIK I.M., Tables of Integrals, Sums, Series, and Products, Nauka, Moscow, 1971.
8. ALEKSANDROV V.M. and KUDISH I.I., Asymptotic analysis of plane and axisymmetric contact problems taking the surface structure of the interacting bodies into account, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 1, 1979.
9. ALEKSANDROV V.M. and KUDISH I.I., Asymptotic methods in contact problems with non-linear friction, Prikl. Mekhan., 17, 6, 1981.
10. GALIN L.A., Contact Problems of Elasticity and Viscoelasticity Theories. Nauka, Moscow, 1980.
11. ALEKSANDROV V.M. and KOVALENKO E.V., Mathematical methods in contact problems with wear. Non-linear Models and Problems of the Mechanics of Deformable Bodies, Nauka, Moscow, 1984.
12. PINEGIN S.V., SHEVELEV I.A. and GUDCHENKO V.M., et al., Influence of External Factors on Contact Strength during Rolling, Nauka, Moscow, 1972.

Translated by M.D.F.